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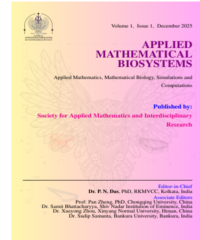

Study of memory effects in a fractional order leslie-gower model with holling type IV functional response

S. Mondal[1]

[1]*Department of Mathematics, Rabindra Mahavidyalaya, Champadanga, Tarakeswar, Hooghly, Pin-712401, West Bengal, India*

Abstract

The use of fractional derivatives in predator–prey models naturally accounts for memory effects, reflecting the fact that population interactions are influenced not only by present conditions but also by their historical states. By incorporating memory through fractional-order operators, these models provide a more realistic description of ecological dynamics and reveal how past population levels can significantly affect stability, persistence, and complex behaviors of predator–prey systems over time. In this paper, following fractional order Caputo derivative approach, here I first convert the integer order three species food chain model to the fractional order model. Some qualitative behaviors of the system like existence and uniqueness, non-negativity and boundedness which are systematically discussed in a feasible region. Local stability criteria of the different equilibrium points have been discussed for fractional order system. Global stability of the interior equilibrium point have been only discussed by defining suitable Lyapunov function. Numerical simulation is performed for different sets of biologically feasible parameter values by using adams-type predictor corrector method (PECE). Numerically it has been observed that the fractional order system shows more complex dynamics, like chaos, bifurcation for lower memory as the fractional order becomes larger and shows more simpler dynamics for higher memory as the order m decreases. Specially, due to memory effect, it becomes stable for lower value of m and the dynamics of the fractional-order system not only depends on system parameters but also depends on fractional order m.

Keywords: *Fractional derivatives and integrals, Food Chain model, Holling type IV functional response, Memory effect, Global stability, Bifurcation, Chaos, Adams-type predictor corrector method (PECE)*

Corresponding author: Shuvojit Mondal *E-mail address:* shuvojitmondal91@gmail.com
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1. INTRODUCTION

Fractional-order integrals and derivatives have wide applications in various fields of science and engineering like physics, chemistry, biology, mathematical sciences and engineering [1–5]. The major reason behind using fractional derivatives is that it has the unique property of capturing the history of the variable, that is, it has memory [6, 7]. Generally, when the output of a system at each time t depends only on the input at time t, such systems are usually known as memory-less system. On the other side, when the system has to remember previous values of the input in order to determine the current value of the output, such systems are known as non-memory less systems, or memory systems [7, 8]. Researchers consider the effect of recent memory as more important than the effect of older memory [9] and such memory effect can not be obtained by the help of integer order derivatives [10, 11]. Therefore, it has been successfully observed the influence of memory concepts on several dynamical systems [8, 12]. It has been recently used in ecological models [13–20] and also in epidemiological models [21, 22]. Generally in past, the study of dynamical systems is described with differential equations where the derivatives were being of integer order. Now by replacing the ordinary time derivative by a fractional order time derivative, a time correlation function or memory kernel appears and therefore the state of the system becomes dependent on all its past states. Therefore, it has been observed that the derivatives with arbitrary order, as introduced by Caputo [23], is a popular choice in modeling real-world systems. The main advantage of using Caputo fractional derivative [23] is that Caputo derivative allows for the use of traditional , making it more intuitive to use in dynamical systems [15, 23]. Any one can observe that in the definition of Caputo fractional derivative, the time correlation function is a power-law function, which has more impact to show the memory effect on dynamical system. To understand the physical interpretation of Caputo fractional derivative deeply, let us recall the definition of Caputo fractional derivative ${}_0^cD_t^m$ of order m defined for an arbitrary function $f(t)$ as follows [23]

$$ {}_0^cD_t^m f(t) = \frac{1}{\Gamma(n-m)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{m-n+1}} d\tau, \quad n-1 \le m < n, \quad (1) $$

where n is an integer, m is a real number. In (1), if $0 < m < 1$ then we can rewrite the equation as follows

$$ {}_0^cD_t^m f(t) = \frac{1}{\Gamma(1-m)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^m} d\tau = \int_0^t w(t-\tau) f^{(1)}(\tau) d\tau, \tag{2} $$

where

$$ w(t) = \frac{t^{-m}}{\Gamma(1-m)}, 0 < m < 1, \tag{3} $$

is the weight function or memory kernel whose task is the storage of system memory over time t [23]. Following (3), I note that the strength of memory is controlled by the order of fractional derivative m. When $m \to 1$, memory of the system decreases and the system tends toward a memory-less system and memory of the system increases with smaller m. Next, I plot the memory kernel function $w(t)$ (Power-law function) with respect to t in $w-t$ plane for different fractional order m $(0 < m < 1)$.

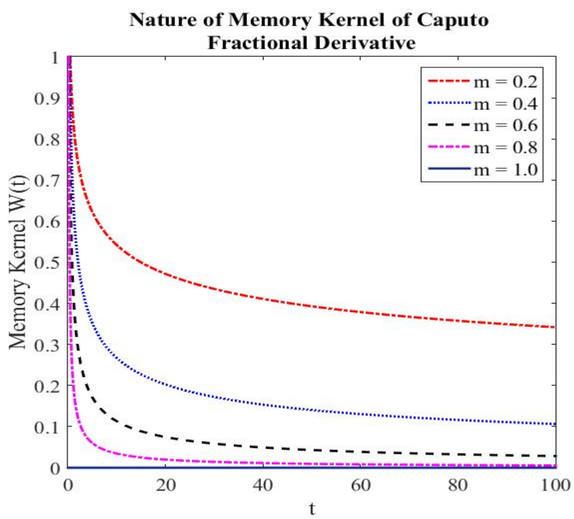

Figure 1. Visualizations of the behavior of the memory kernel function $w(t)$ (Power function) with respect to t in $w - t$ plane has been presented for different fractional order m $(0 < m < 1)$ and also for integer order $m = 1$.

Since the decaying rate of this type of memory kernel occurring in Caputo fractional derivative depends on fractional order m and it is increasing with the decreasing of fractional order m, thus Caputo fractional derivative is widely used to study the memory effect of fractional order model. Therefore the Caputo fractional derivative, involving a convolution integral with a power-law memory kernel, are useful to describe memory effects in dynamical systems.

In population models, chaos is especially intriguing for researchers as it explains how simple biological interactions can lead to complex, irregular, and unpredictable fluctuations in species densities over time. After a fascinating research by May [24], exploring the chaotic behaviors in population models became a fascination for many researchers in recent times. Lots of mathematical models have been proposed on the basis of food-chain and analyzed to show complicated dynamics like chaos, bifurcation etc. [25–31]. Aziz-Alaoui [32] discussed the complex dynamics in a modified Leslie-Gower three species food chain model with Holling type II response function. Again, Alidousti and Ghahfarokhi [6] extended the work of Aziz-Alaoui [32] and analyzed the following fractional order tri-trophic food chain model with Holling type II functional response. Sambath in [33] also studied the asymptotic behavior of a fractional order three species predator-prey model with the same functional response. In recent past, Saeedian et.al. [21] studied the evolution of the fractional order SIR epidemic model by considering memory effects. Using the fractional calculus technique, Saeedian et.al. shown that the dynamics of such a system depends on the strength of memory effects, controlled by the order of fractional derivatives. Investigation about the influence of memory effect in a fractional order Leslie-Gower type model with Holling type IV functional response is relatively less studied in population ecology. Therefore, in this paper, I consider a three species food chain model with simplified Holling type IV functional response to understand underlying dynamics of the model with memory effect respect to fractional order.

Here, I consider an integer order three species food chain model which was studied by Ali et. al. [34]. They actually investigated the following three-dimension coupled nonlinear autonomous system of integer order differential equations with non-monotone functional response (also called simplified Holling type IV functional response) to understand the underlying dynamics of food chain model:

$$ \begin{aligned} \frac{dX}{dT} &= a_0X - b_0X^2 - \frac{v_0XY}{d_1+X^2}, \ X(0) \ge 0, \\ \frac{dY}{dT} &= -a_1Y + \frac{v_1XY}{d_1+X^2} - \frac{v_2YZ}{d_2+Y}, \ Y(0) \ge 0, \\ \frac{dZ}{dT} &= Z\Big(c_3Z - \frac{v_3Z}{d_3+Y}\Big), \ Z(0) \ge 0, \end{aligned} \tag{4} $$

where X, Y, Z are, respectively, the densities of prey, intermediate predator and top predator at any instant of at time T. This model considers interactions between a generalist top predator, specialist middle predator, and prey. Here, the specialist middle predator is consumed by the top predator, at a Holling type II rate. The interactions between the specialist middle predator and prey are modeled via a modified Holling type IV functional response. The interaction between the generalist top predator and specialist middle predator follow a modified Leslie-Gower scheme. That is the generalist top predator grows quadratically, because of sexual reproduction as c_3Z^2, and loses because of intra-species competition as $-\frac{v_3Z^2}{d_3+Y}$. The d_3 signifies that Z is a generalist. The biological interpretation of all the parameters are described in the following table.

Table 1. Parameter interpretation

Symbol	Meaning
a_0	Growth rate of prey
b_0	Intra specific competition coefficient
$v_i's$	Maximum values that per-capita rate can attain
d_1	Measure of protection level provided by the environment to the prey
a_1	Death rate of intermediate predator
d_2	Half-Saturation constant
c_3	Growth rate of top predator via sexual reproduction
d_3	Residual loss of top predator due to severe scarcity of it's favorite prey, Y

All parameters are non-zero positive. For description of the model and system parameters, readers are referred to [32, 34].

The rest of this paper is organized as follows: in Section 2, following fractional order Caputo derivative approach, I convert the integer order differential equations of the three species predator-prey model (4) to the fractional order differential equations, thereby allowing us to consider memory effects. Mathematical results related to qualitative behaviors of the system (10) like existence, uniqueness, non-negativity and boundedness will be investigated systematically in Section 3. In section 4, Local stability criteria of all feasible equilibrium points have been discussed for fractional order system. Global stability of the interior equilibrium point have been only discussed here. Using parameter values, numerically it has been observed that the fractional order system (10) shows more complex dynamics, like chaos, bifurcation for lower memory as the fractional order becomes larger and shows more simpler dynamics for higher memory as the order m decreases. Specially, due to memory effect, it becomes stable for lower value of m. Simulation results are also given to validate the analytical results in Section 5 .

2. Memorial Process from Integer order system to Fractional order system

To observe the influence of memory effect, first I rewrite the equation (4) in terms of time-dependent integral as [34]:

$$\begin{aligned}\frac{dX}{dT} &= \int_0^T w(T-\tau)\left(a_0X(\tau)-b_0X^2(\tau)-\frac{\upsilon_0X(\tau)Y(\tau)}{X^2(\tau)+d_0}\right)d\tau,\\ \frac{dY}{dT} &= \int_0^T w(T-\tau)\left(\frac{\upsilon_1X(\tau)Y(\tau)}{X^2(\tau)+d_1}-a_1Y(\tau)-\frac{\upsilon_2Y(\tau)Z(\tau)}{Y(\tau)+d_2}\right)d\tau,\\ \frac{dZ}{dT} &= \int_0^T w(T-\tau)\left(c_3Z^2(\tau)-\frac{\upsilon_3Z^2(\tau)}{Y(\tau)+d_3}\right)d\tau, \qquad (5)\end{aligned}$$

in which $w(T-\tau)$ plays the role of time-dependent kernel. A power-law function is a suitable choice to explain the long-term memory effects which exhibits a slow decay such that the state of the system at quite early times also contributes to the evolution of the system. This type of kernel ensures the existence of memory effect whenever a system contains the fractional order derivative. Thus, to generate the fractional order model I consider the following power-law correlation function for $w(T-\tau)$:

$$w(T-\tau)=\frac{1}{\Gamma(m-1)}(T-\tau)^{(m-2)},\ 0<m<1, \qquad (6)$$

in which $\Gamma(x)$ is the well known gamma function. Note that $\Gamma(x)$ is well defined for negative non-integer values [23]. The choice of such kernel allow us to rewrite the equation (5) to the form of fractional differential equations with Caputo-type derivative. Now this kernel is substituted into the first equation of (5) and I get

$$\begin{aligned}\frac{dX}{dT} &= \int_0^T \frac{(T-\tau)^{(m-2)}}{\Gamma(m-1)}\left(a_0X(\tau)-b_0X^2(\tau)-\frac{\upsilon_0X(\tau)Y(\tau)}{X^2(\tau)+d_0}\right)d\tau,\\ DX(T) &= {}_0I_T^{(m-1)}\left[a_0X(\tau)-b_0X^2(\tau)-\frac{\upsilon_0X(\tau)Y(\tau)}{X^2(\tau)+d_0}\right], \qquad (7)\end{aligned}$$

where $D\equiv\frac{d}{dT}$ is the integer order differential operator and ${}_0I_T^{(m-1)}$ is the fractional integral of order $(m-1)$ on the interval $[0,T]$. Next I apply a Caputo fractional derivative of order $(m-1)$ on both sides of (7) which is denoted by ${}_0^cD_T^{(m-1)}$ and I observe

$$\begin{aligned}&{}_0^cD_T^{(m-1)}D\,X(T)=\\ &{}_0^cD_T^{(m-1)}\,{}_0I_T^{(m-1)}\left[a_0X(\tau)-b_0X^2(\tau)-\frac{\upsilon_0X(\tau)Y(\tau)}{X^2(\tau)+d_0}\right]. \qquad (8)\end{aligned}$$

Now using the fact that the Caputo fractional derivative and fractional integral are inverse operators [7], i.e., for some differentiable function $f(t)$, ${}_0^cD_t^n\,{}_0I_t^nf=f$, for any real $n\geq 0$, I obtain the following fractional differential equation from (8)

$${}_0^cD_T^mX(T) = a_0X(T)-b_0X^2(T)-\frac{\upsilon_0X(T)Y(T)}{X^2(T)+d_0}.$$

Similarly one can obtain from the last two equations of (5)

$$\begin{aligned}{}_0^cD_T^mY(T) &= \frac{\upsilon_1X(T)Y(T)}{X^2(T)+d_1}-a_1Y(T)-\frac{\upsilon_2Y(T)Z(T)}{Y(T)+d_2},\\ {}_0^cD_T^mZ(T) &= c_3Z^2(T)-\frac{\upsilon_3Z^2(\tau)}{Y(T)+d_3}.\end{aligned}$$

Therefore our integer order model (4) reduces to fractional order model

$$\begin{aligned}{}_0^cD_T^mX &= a_0X-b_0X^2-\frac{\upsilon_0XY}{d_1+X^2},\\ {}_0^cD_T^mY &= -a_1Y+\frac{\upsilon_1XY}{d_1+X^2}-\frac{\upsilon_2YZ}{d_2+Y}, \qquad (9)\\ {}_0^cD_T^mZ &= Z\Big(c_3Z-\frac{\upsilon_3Z}{d_3+Y}\Big).\end{aligned}$$

For simple mathematical calculation, I use the following transformations

$$X=\frac{a_0}{b_0}x,\ Y=\frac{a_0^2}{b_0\upsilon_0}y,\ Z=\frac{a_0^3}{b_0\upsilon_0\upsilon_2}z,\ T=\frac{t}{a_0},$$

the system (9) takes the simplified form

$$\begin{aligned}{}_0^cD_t^mx &= x(1-x)-\frac{sxy}{x^2+a},\ x(0)=x_0\geq 0,\\ {}_0^cD_t^my &= \frac{cxy}{x^2+a}-by-\frac{yz}{y+d},\ y(0)=y_0\geq 0, \qquad (10)\\ {}_0^cD_t^mz &= pz^2-\frac{qz^2}{y+r},\ z(0)=z_0\geq 0,\end{aligned}$$

where

$$a=\frac{b_0^2d_1}{a_0^2},\ b=\frac{a_1}{a_0},\ c=\frac{b_0\upsilon_1}{a_0^2},\ d=\frac{d_2\upsilon_0b_0}{a_0^2},\ p=\frac{c_3a_0^2}{b_0\upsilon_0\upsilon_2},$$

$$q=\frac{\upsilon_3}{\upsilon_2},\ r=\frac{d_3\upsilon_0b_0}{a_0^2},\ s=\frac{b_0}{a_0}.$$

with the initial conditions $x(0)\geq 0, y(0)\geq 0, z(0)\geq 0$, where ${}_0^cD_t^m$ is the Caputo fractional derivative of order m. The state space of the system (10) is the positive cone $\mathbb{R}_+^3=\{(x,y,z)\in\mathbb{R}^3 : x\geq 0, y\geq 0, z\geq 0\}$.

3. Existence, non-negativity and boundedness

Here I study the existence and uniqueness of the solution of our system (10) first in this section. In recent years, some researchers proved the positivity of solutions before discussing the existence and uniqueness of a dynamical system [35]. Ghani et. al. [36]

also established boundedness of solutions even before proving the existence and non-negativity of it. Roy et. al. [37] followed a systematical apporach in their reaserch work and they proved the existence and uniqueness of the solutions first while establishing qualitative behaviour of solutions of a system. Later, they discussed about non-negativity, boundedness and positivity of the solutions which I find more interesting fact than the previous studies. In a similar manner, I shall also follow a systematic order to establish the existence and uniqueness of solutions of the model (10), followed by non-negativity, boundedness which makes this section more interesting compared to other existing literature.

3.1. Existence and uniqueness

To study the existence and uniqueness of the solution of our system (10), I have the following Lemma due to Li et al [38].

Lemma 1 *[38] Consider the system*

$$ {}_{t_0}^{c}D_t^m x(t) = f(t,x), t > t_0 $$

with initial condition x_{t_0}, *where* $0 < m \leq 1$, $f : [t_0, \infty) \times A \to \mathbb{R}^n$, $A \in \mathbb{R}^n$. *If* $f(t,x)$ *satisfies the locally Lipschitz condition with respect to* x *then there exists a unique solution of the above system on* $[t_0, \infty) \times A$.

Using Lemma (1), here I study the existence and uniqueness of the solution of system (10) in the region $A \times [0, T_1]$, where $A = \{(x,y,z) \in \mathbb{R}^3 | \; max\{|x|, |y|, |z|\} \leq M_1\}$, $T_1 < \infty$ and M_1 is large. Denote $X = (x,y,z)$, $\bar{X} = (\bar{x}, \bar{y}, \bar{z})$. Consider a mapping $H : A \to \mathbb{R}^3$ such that $H(X) = (H_1(X), H_2(X), H_3(X))$, where

$$
\begin{aligned}
H_1(X) &= x(1-x) - \frac{sxy}{x^2+a}, \\
H_2(X) &= \frac{cxy}{x^2+a} - by - \frac{yz}{y+d}, \qquad (11)\\
H_3(x) &= pz^2 - \frac{qz^2}{y+r}.
\end{aligned}
$$

For any $X, \bar{X} \in A$, it follows from (11) that

$$
\begin{aligned}
&\| H(X) - H(\bar{X}) \| \\
&= | H_1(X) - H_1(\bar{X}) | + | H_2(X) - H_2(\bar{X}) | \\
&+ | H_3(X) - H_3(\bar{X}) | \\
&= | x(1-x) - \frac{sxy}{x^2+a} - \bar{x}(1-\bar{x}) + \frac{s\bar{x}\bar{y}}{\bar{x}^2+a} | \\
&+ | \frac{cxy}{x^2+a} - by - \frac{yz}{y+d} - \frac{c\bar{x}\bar{y}}{\bar{x}^2+a} + b\bar{y} + \frac{\bar{y}\bar{z}}{\bar{y}+d} | \\
&+ | pz^2 - \frac{qz^2}{y+r} - p\bar{z}^2 + \frac{q\bar{z}^2}{\bar{y}+r} | \\
&= | (x-\bar{x}) - (x^2-\bar{x}^2) - s\Big(\frac{xy}{x+a} - \frac{\bar{x}\bar{y}}{\bar{x}+a}\Big) | \\
&+ | c\Big(\frac{xy}{x+a} - \frac{\bar{x}\bar{y}}{\bar{x}+a}\Big) - b(y-\bar{y}) - (\frac{yz}{y+d} - \frac{\bar{y}\bar{z}}{\bar{y}+d}) | \\
&+ | p(z^2-\bar{z}^2) - q(\frac{z^2}{y+r} - \frac{\bar{z}^2}{\bar{y}+r}) |
\end{aligned}
$$

$$
\begin{aligned}
&\leq | x-\bar{x} | + | x^2-\bar{x}^2 | \\
&+ (s+c) \,|\, \Big(\frac{xy}{x+a} - \frac{\bar{x}\bar{y}}{\bar{x}+a}\Big) | + b | y-\bar{y} | \\
&+ | \frac{yz}{y+d} - \frac{\bar{y}\bar{z}}{\bar{y}+d} | + p | z^2-\bar{z}^2 | + q | (\frac{z^2}{y+r} - \frac{\bar{z}^2}{\bar{y}+r}) | \\
&\leq | x-\bar{x} | + 2M_1 | x-\bar{x} | + \frac{(s+c)}{a^2} | a(xy-\bar{x}\bar{y}) \\
&+ x\bar{x}(y-\bar{y}) | + b | y-\bar{y} | + \frac{1}{d^2} | d(yz-\bar{y}\bar{z}) + y\bar{y}(z-\bar{z}) | \\
&+ p | z^2-\bar{z}^2 | + \frac{q}{r^2} | r(z^2-z^2) + (z^2\bar{y} - \bar{z}^2 y) | \\
&\leq \Big(1 + 2M_1 + \frac{M_1(s+c)}{a}\Big) | x-\bar{x} | \\
&+ \Big(M_1(\frac{s+c}{a} + \frac{1}{d}) + b + M_1^2(\frac{(s+c)}{a^2} + \frac{q}{r^2})\Big) | y-\bar{y} | \\
&+ \Big(M_1(2p + \frac{1}{d} + \frac{2q}{r}) + M_1^2(\frac{1}{d^2} + \frac{2q}{r^2})\Big) | z-\bar{z} | \\
&\leq L \| (x,y,z) - (\bar{x},\bar{y},\bar{z}) \| \\
&\leq L \| X - \bar{X} \|,
\end{aligned}
$$

where $L = max\{1 + 2M_1 + \frac{M_1(s+c)}{a}, M_1(\frac{s+c}{a} + \frac{1}{d}) + b + M_1^2(\frac{(s+c)}{a^2} + \frac{q}{r^2}), M_1(2p + \frac{1}{d} + \frac{2q}{r}) + M_1^2(\frac{1}{d^2} + \frac{2q}{r^2})\}$.

Thus $H(X)$ satisfies Lipschitz condition with respect to X and following Lemma (1), there exists a unique solution $X(t)$ of the system (10) with the initial condition $X(0) = (x(0), y(0), z(0))$.

3.2. Non-negativity

Considering the biological significance of the model, I am only interested in solutions that are non-negative and bounded in the region $\mathbb{R}_+^3 = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$. To prove the non-negativity and uniform boundedness of our system, I shall use the following lemmas.

Lemma 2 *[39] Suppose that* $f(t) \in C[a,b]$ *and* $D_a^m f(t) \in C(a,b]$ *with* $0 < m \leq 1$. *The Generalized Mean Value Theorem states that*

$$ f(t) = f(a) + \frac{1}{\Gamma(m)} (D_a^m f)(\xi).(t-a)^m, $$

where $a \leq \xi \leq t$, $\forall t \in (a,b]$.

Lemma 3 *[40] Let* $u(t)$ *be a continuous function on* $[t_0, \infty)$ *and satisfying*

$$ {}_{t_0}^{c}D_t^m u(t) \leq -\lambda u(t) + \mu, $$
$$ u(t_0) = u_{t_0}, $$

where $0 < m \leq 1$, $(\lambda, \mu) \in \mathbb{R}^2$, $\lambda \neq 0$ *and* $t_0 \geq 0$ *is the initial time. Then its solution has the form*

$$ u(t) \leq \Big(u_{t_0} - \frac{\mu}{\lambda}\Big) E_m[-\lambda(t-t_0)^m] + \frac{\mu}{\lambda}. $$

We have the following existence results on initial value problem (IVP) with caputo type fractional order differential equations.

Lemma 4 *[41] Consider the initial value problem (IVP) with caputo type FDE*

$$ {}_{t_0}^{c}D_t^m x(t) = f(t, x(t)), \;\; x(0) = x_0 \qquad (12) $$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $0 < m < 1$. *Assume that* $f \in C(R_0, R)$, *where* $R_0 = [(t,x) : 0 \leq t \leq a, | x - x_0 | \leq b]$ *and let* $| f(t,x) | \leq N$ *on* R_0. *Then there exists at least one solution for the IVP (12) on* $0 \leq t \leq \gamma$ *where* $\gamma = min\Big(a, [\frac{b}{M}\Gamma(m+1)]^{\frac{1}{m}}\Big)$, $0 < m < 1$.

Lemma 5 *[41] Consider the initial value problem (IVP) given by (12). Let*

$$g(v, x_*(v)) = f(t-(t^m - v\Gamma(m+1))^{\frac{1}{m}},\ x(t-(t^m - v\Gamma(m+1))^{\frac{1}{m}}))$$

and assume that the conditions of lemma (4) hold. Then, a solution of (12), is given by

$$x(t) = x_*(\frac{t^m}{\Gamma(m+1)})$$

where $x_*(v)$ is the solution of the integer order differential equation

$$\frac{d(x_*(v))}{dv} = g(v, x_*(v))$$

with the initial condition $x_*(0) = x_0$.

Theorem 1 *All the solutions of system (10) which start in $\mathbb{R}^3_+$ are non-negative.*

Proof: First I show that the solutions $x(t) \in \mathbb{R}^3_+$ are non negative if it starts with positive initial values. If not, then there exists a $t_1 > 0$ such that

$$\begin{aligned} x(t) &> 0,\ \ 0 \le t < t_1, \\ x(t) &= 0,\ \ t = t_1, \\ x(t_1^+) &< 0. \end{aligned} \tag{13}$$

Using (13) in the first equation of (10), we have

$${}_0^cD_t^m x(t)|_{t=t_1} = 0. \tag{14}$$

According to Lemma (2), I have $x(t_1^+) = 0$, which contradicts the fact $x(t_1^+) < 0$. Therefore, I have $x(t) \ge 0,\ \forall\, t \ge 0$. Using similar arguments, one can prove $y(t) \ge 0, \forall t \ge 0$ and $z(t) \ge 0, \forall t \ge 0$. So, it is proven that all the solutions of system (10) which start in $\mathbb{R}^3_+$ are non negative.

3.3. Boundedness

Next I will show that, under some assumptions, all solutions $x(t), y(t)$ and $z(t)$ of our system (10) are uniformly bounded for sufficiently large t.

Theorem 2 *All the non negative solutions of system (10) which are initiating in $\mathbb{R}^3_+$ are uniformly bounded, provided*

$$\beta + \frac{\beta}{4b} + r < \frac{q}{p} \tag{15}$$

and ultimately entering the region

$$\Omega = \Big\{(x,y,z) \in \mathbb{R}^3_+ : 0 \le x \le 1, 0 \le x + \frac{y}{\beta} \le 1 + \frac{1}{4b},$$

$$0 \le x + \frac{y}{\beta} + \alpha z \le 1 + \frac{1}{4b} + \frac{M}{b}\Big\},$$

where

$$\beta = \frac{v_1}{a_0},\ \alpha = \frac{1}{b^2(\beta + \frac{\beta}{4b} + r)},\ M = \frac{1}{4(q - (\beta + \frac{\beta}{4b} + r)p)}.$$

Proof: (i) Let $(x(0), y(0), z(0)) \in \Omega$ and from the following Theorem (1), $(x(t), y(t), z(t))$ remain non negative in $\mathbb{R}^3_+$, then I will show that $(x(t), y(t), z(t)) \in \Omega, \forall t \ge 0$ and $\forall m \in (0,1]$. Now to reach our main objective, I have to prove the following steps for all $t \ge 0$ and $\forall m \in (0,1]$,

Step (i-a): $x(t) \le 1$;

Step (i-b): $x(t) + \frac{y(t)}{\beta} \le 1 + \frac{1}{4b}$;

Step (i-c): $x(t) + \frac{y(t)}{\beta} + \alpha z(t) \le 1 + \frac{1}{4b} + \frac{M}{b}$.

Proof of Step (i-a): I first prove that $x(t) \le 1, \forall t \ge 0$ and $\forall m \in (0,1]$. Since $x \ge 0, y \ge 0, z \ge 0$ in $\mathbb{R}^3_+$, then any solution $\phi(t) = (x(t), y(t), z(t))$ of (10), which starts in $\mathbb{R}^3_+$, must satisfy the fractional order differential inequation

$${}_0^cD_t^m x \le x(1-x), x(0) = x_0 > 0, m \in (0,1], \tag{16}$$

which is clearly obtained from the first equation of (10). Moreover, this equation (16) represents a fractional order logistic equation. Now I can apply Lemma (5) to solve this fractional order differential inequation (16). Here

$$g(v, x_*(v)) = x_*(v)(1 - x_*(v))$$

Then corresponding integer order differential in-equation of this fractional IVP (16) is

$$\frac{d(x_*(v))}{dv} \le x_*(v)(1 - x_*(v)),\ \ x_*(0) = x_0.$$

The solution of this integer order linear IVP is

$$x_*(v) \le \frac{1}{1 + c_1 e^{-v}},$$

where $c_1 = \frac{1}{x_0} - 1$. Consequently, the solution of the given fractional order IVP (16) is

$$x(t) = x_*(\frac{t^m}{\Gamma(m+1)}) \le \frac{1}{1 + c_1 e^{-\frac{t^m}{\Gamma(m+1)}}} \le 1, \forall t \ge 0,\ 0 < m \le 1.$$

Then any solution $x(t)$ of (16) must be bounded by 1 with respect to any fractional order $m \in (0,1]$. Therefore, it follows that any non negative solution $\phi(t)$ of (10) satisfies $x(t) \le 1, \forall t \ge 0$ and $\forall m \in (0,1]$.

Proof of Step (i-b): I now prove that $x(t) + \frac{y(t)}{\beta} \le 1 + \frac{1}{4b}, \forall t \ge 0$ and $\forall m \in (0,1]$. Let us define a function

$$V_1(t) = x(t) + \frac{y(t)}{\beta}, \tag{17}$$

Taking fractional time derivative, I have

$${}_0^cD_t^m V_1(t) = {}_0^cD_t^m x(t) + {}_0^cD_t^m \frac{y(t)}{\beta} = x(1-x) - \frac{b}{\beta} y - \frac{1}{\beta}\frac{yz}{y+d}.$$

Since all parameters are positive and solutions initiating in $\mathbb{R}^3_+$ then,

$$\begin{aligned} {}_0^cD_t^m V_1(t) &\le x(1-x) - \frac{b}{\beta} y, \\ {}_0^cD_t^m V_1(t) &\le x(1-x) + bx - b(x + \frac{y}{\beta}), \\ {}_0^cD_t^m V_1(t) &+ bV_1(t) \le b + \frac{1}{4}, \end{aligned}$$

since in Ω, $0 \le x \le 1$ and $max_{[0,1]} x(1-x) = \frac{1}{4}$. Applying Lemma

(3), I have

$$V_1(t) \le (V_1(0) - (1+\frac{1}{4b}))E_m[-bt^m] + (1+\frac{1}{4b}), \\ = V_1(0)E_m[-bt^m] + (1+\frac{1}{4b})(1-E_m[-bt^m]). \tag{18}$$

For $t \to \infty$, we thus have $V_1(t) \to (1+\frac{1}{4b})$. Therefore, $V_1(t) \le (1+\frac{1}{4b}), \forall t \ge 0$ and $\forall m \in (0,1]$. Hence it follows that any non negative solution of (10) satisfies $x(t) + \frac{y(t)}{\beta} \le 1 + \frac{1}{4b}, \forall t \ge 0$ and $\forall m \in (0,1]$.

Proof of Step (i-c): I finally prove that $x(t) + \frac{y(t)}{\beta} + \alpha z(t) \le 1 + \frac{1}{4b} + \frac{M}{b}, \forall t \ge 0$ and $\forall m \in (0,1]$ holds, with

$$\alpha = \frac{1}{b^2(\beta + \frac{\beta}{4b} + r)},\ M = \frac{1}{4(q - (\beta + \frac{\beta}{4b} + r)p)},$$

provided $\beta + \frac{\beta}{4b} + r < \frac{q}{p}$. Again I define a function

$$V_2(t) = x(t) + \frac{y(t)}{\beta} + \alpha z(t), \tag{19}$$

Taking fractional order time derivative, I have

$${}_0^cD_t^m V_2(t) = {}_0^cD_t^m x(t) + {}_0^cD_t^m \frac{y(t)}{\beta} + {}_0^cD_t^m \alpha z(t), \\ = x(1-x) - \frac{b}{\beta}y - \frac{1}{\beta}\frac{yz}{y+d} + \alpha(p - \frac{q}{y+r})z^2.$$

Similarly to the previous step (i-b), since all parameters are positive, all solutions initiating in $Int(\mathbb{R}^3_+)$ remain non negative and in Ω, $0 \le x \le 1$, $max_{[0,1]}x(1-x) = \frac{1}{4}, y \le \beta + \frac{\beta}{4b}$, I get

$$\begin{aligned} {}_0^cD_t^m V_2(t) \le & \quad \tfrac{1}{4} + b - bV_2(t) + \alpha bz + \alpha(p - \tfrac{q}{y+r})z^2, \\ {}_0^cD_t^m V_2(t) \le & \quad \tfrac{1}{4} + b - bV_2(t) + \alpha bz + \alpha(p - \tfrac{q}{\beta + \frac{\beta}{4b} + r})z^2, \\ {}_0^cD_t^m V_2(t) + & \quad bV_2(t) \le b + \tfrac{1}{4} + M, \end{aligned} \tag{20}$$

where,

$$M = max_{z\in\mathbb{R}_+}\left(\alpha bz + \alpha(p - \frac{q}{\beta + \frac{\beta}{4b} + r})z^2\right).$$

Now I intend to find M, the maxima for the function $f(z) = \alpha bz + \alpha(p - \frac{q}{\beta+\frac{\beta}{4b}+r})z^2, z \in \mathbb{R}_+$. Here $f'(z) = \alpha b + 2\alpha z(p - \frac{q}{\beta+\frac{\beta}{4b}+r})$ and $f''(z) = 2\alpha(p - \frac{q}{\beta+\frac{\beta}{4b}+r})$. Since $\beta + \frac{\beta}{4b} + r < \frac{q}{p}$, I observe $f''(z) < 0$ and hence $max[f(z)]$ exists at

$$z = \frac{b}{2(\frac{q}{\beta+\frac{\beta}{4b}+r} - p)} = z_1(say).$$

Therefore using $\alpha = \frac{1}{b^2(\beta+\frac{\beta}{4b}+r)}$, I have

$$M = max[f(z)]\ |_{z=z_1} = \frac{\alpha b^2(\beta + \frac{\beta}{4b} + r)}{4(q - (\beta + \frac{\beta}{4b} + r)p)} = \frac{1}{4(q - (\beta + \frac{\beta}{4b} + r)p)}. \tag{21}$$

Then applying Lemma (3) on (20), I have

$$V_2(t) \le (V_2(0) - (1 + \frac{1}{4b} + \frac{M}{b}))E_m[-bt^m] + (1 + \frac{1}{4b} + \frac{M}{b}), \\ = V_2(0)E_m[-bt^m] + (1 + \frac{1}{4b} + \frac{M}{b})(1 - E_m[-bt^m]). \tag{22}$$

For $t \to \infty$, we thus have $V_2(t) \to (1 + \frac{1}{4b} + \frac{M}{b})$. Therefore, $V_2(t) \le (1 + \frac{1}{4b} + \frac{M}{b}), \forall t \ge 0$ and $\forall m \in (0,1]$. Hence it follows that any non negative solution of (10) satisfies for all $m \in (0,1]$,

$$\begin{aligned} & x(t) + \frac{y(t)}{\beta} + \frac{1}{b^2(\beta + \frac{\beta}{4b} + r)}z(t) \\ \le \quad & 1 + \frac{1}{4b} + \frac{1}{b}\frac{1}{4(q - (\beta + \frac{\beta}{4b} + r)p)}, \quad \forall t \ge 0. \end{aligned}$$

Therefore, all the non negative solutions $x(t), y(t), z(t)$ of system (10) initiating in $\mathbb{R}^3_+$ are uniformly bounded and entering the set Ω, $\forall t \ge 0$ and $\forall m \in (0,1]$.

4. Existence and stability of equilibria

I have the following stability result on fractional order dynamical systems.

Theorem 3 *[20, 42, 43] Consider the following fractional order system*

$${}_0^cD_t^m x(t) = f(x), x(0) = x_0$$

with $0 < m < 1, x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. The equilibrium points of the above system are calculated by solving the equation: $f(x) = 0$. These equilibrium points are locally asymptotically stable if all eigenvalues λ_i of the Jacobin matrix $J = \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy $|\ Arg(\lambda_i)\ | > \frac{m\pi}{2}, i = 1, 2, ...x, n$.

An equilibrium point of system (10) is found by solving the three equations $D_t^m x(t) = D_t^m y(t) = D_t^m z(t) = 0$ in (10). There are four biologically feasible non-negative equilibrium points of system (10). The trivial equilibrium $E_0 = (0,0,0)$ and the axial equilibrium $E_1 = (1,0,0)$ are always exist. The planner equilibrium point $E_2 = (\bar{x}, \bar{y}, 0)$ exists uniquely in the positive quadrant of $xy-$ plane, where $\bar{x}$ and $\bar{y}$ are given by

$$\bar{x} = \frac{c}{2b},\ \bar{y} = \frac{1}{s}(1 - \bar{x})(a + \bar{x}^2),$$

provided that the following conditions are hold:

$$\frac{c}{2b} < 1,\ c^2 - 4ab^2 = 0. \tag{23}$$

I observe that in the absence of prey x, both predators y and z can not survive. So there is no equilibrium point in the $yz-$ plane. Similarly I can also conclude that there is no equilibrium point in $xz-$ plane. Now there exists a unique interior equilibrium point $E^* = (x^*, y^*, z^*)$ of the system (3), where the equilibrium population densities are given by

$$y^* = \frac{q}{p} - r, \tag{24}$$

while x^* is the positive root of the cubic equation

$$x^3 - x^2 + ax + (sy^* - a) = 0, \tag{25}$$

this equation can be written as

$$f(x) = Ax^3 + Bx^2 + Cx + D = 0, \tag{26}$$

where $A = 1, B = -1, C = a$ and $D = (sy^* - a)$. Now since $0 \le x^* \le 1$, then $f(0) = D < 0$, if $y^* < \frac{a}{s}$ and $f(1) = sy^* > 0$. Thus, $f(0)f(1) = sy^*(sy^* - a) < 0$, and then there is a positive root of equation (26) lies in (0, 1) when $y^* < \frac{a}{s}$ is satisfied. Now from the second equation of system (10), I obtain

$$z^* = (-b + \frac{cx^*}{a + x^{*2}})(y^* + d), \tag{27}$$

and it exists if $b < \frac{cx^*}{a+x^{*2}}$. Therefore the positivity condition of E^* in $\mathbb{R}^3_+$ are

$$y^* < \frac{a}{s},\ b < \frac{cx^*}{a + x^{*2}},$$

where $v_3 > c_3 d_3$. Different stability results for the equilibrium points E_0, E_1, E_2 and E^* are given in the following.

Now to investigate the dynamical behavior of the equilibrium points E_i, $(i = 0, 1, 2)$ and E^*, I first construct the Jacobian matrix J evaluated at an equilibrium point (x, y, z) of the system (10) is

$$J(x, y, z) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{28}$$

where $a_{11} = 1 - 2x - \frac{sy(a-x^2)}{(a+x^2)^2}$, $a_{12} = -\frac{sx}{a+x^2}$, $a_{13} = 0$, $a_{21} = \frac{cy(a-x^2)}{(a+x^2)^2}$,
$a_{22} = -b + \frac{cx}{a+x^2} - \frac{dz}{(d+y)^2}, a_{23} = -\frac{y}{d+y}$, $a_{31} = 0$, $a_{32} = \frac{qz^2}{(r+y)^2}$,
$a_{33} = 2z(p - \frac{q}{y+r})$.

Then the Jacobian matrices evaluated at E_0, E_1 and E_2 are given by

$$J(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$J(E_1) = \begin{pmatrix} -1 & -\frac{s}{a+1} & 0 \\ 0 & -b + \frac{c}{a+1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$J(E_2) = \begin{pmatrix} 1 - 2\bar{x} - \frac{(1-\bar{x})(a-\bar{x}^2)}{a+\bar{x}^2} & -\frac{s\bar{x}}{a+\bar{x}^2} & 0 \\ \frac{c(1-\bar{x})(a-\bar{x}^2)}{s(a+\bar{x}^2)} & 0 & -\frac{\bar{y}}{(d+\bar{y})} \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, the eigenvalues of $J(E_0)$ are $\xi_1 = 1$, $\xi_2 = -b$ and $\xi_3 = 0$. Note that $Arg(\xi_3)$ is undefined. Since one them is a positive real and anothe one is a negative real, then E_0 is always unstable. Therefore E_0 is non-hyperbolic.

Next, the eigenvalues of $J(E_1)$ are $\xi_1 = -1\ (< 0)$, $\xi_2 = \frac{c-b-ab}{1+a}$ and $\xi_3 = 0$. Hence E_1 is also non-hyperbolic. Note that If $c - b > ab$ then $\xi_2 > 0$. In this case, E_1 is always unstable saddle along $x-$ direction. If $c - b < ab$ then $\xi_2 < 0$. Consequently, two of the eigenvalues are negative real, so in this case E_1 is stable manifold along x and $y-$ direction.

Again from the variational matrix of E_2, the eigenvalues of $J(E_2)$ are $\xi_{1,2} = \frac{1}{2}[P \pm \sqrt{P^2 - 4Q}]$, where $P = 1 - 2\bar{x} - \frac{(1-\bar{x})(a-\bar{x}^2)}{a+\bar{x}^2}$, $Q = \frac{c\bar{x}(1-\bar{x})(a-\bar{x}^2)}{(a+\bar{x}^2)^2}$ and $\xi_3 = 0$. Since one of the eigenvalue ξ_3 becomes zero, so E_2 is non-hyperbolic equilibrium point.

For local stability of the interior equilibrium E^*, I compute the Jacobian matrix of system (10) at $E^* = (x^*, y^*, z^*)$ as

$$J(E^*) = \begin{pmatrix} 1 - 2x^* - \frac{(1-x^*)(a-x^{*2})}{a+x^{*2}} & -\frac{sx^*}{a+x^{*2}} & 0 \\ \frac{c(1-x^*)(a-x^{*2})}{s(a+x^{*2})} & \frac{y^*z^*}{(y^*+d)^2} & -\frac{y^*}{d+y^*} \\ 0 & \frac{pz^{*2}}{y^*+r} & 0 \end{pmatrix}. \tag{29}$$

The eigenvalues are the roots of the cubic equation

$$F(\xi) = \xi^3 + A_1\xi^2 + A_2\xi + A_3 = 0, \tag{30}$$

where $A_1 = -1 + 2x^* + \frac{(1-x^*)(a-x^{*2})}{a+x^{*2}} - \frac{y^*z^*}{(y^*+d)^2}$,
$A_2 = \frac{y^*z^*}{(y^*+d)^2}\left(1 - 2x^* - \frac{(1-x^*)(a-x^{*2})}{a+x^{*2}}\right) + \frac{cx^*(1-x^*)(a-x^{*2})}{(a+x^{*2})^2} + \frac{py^*z^{*2}}{(y^*+d)(y^*+r)}$,
$A_3 = -\frac{py^*z^{*2}}{(y^*+d)(y^*+r)}\left(1 - 2x^* - \frac{(1-x^*)(a-x^{*2})}{a+x^{*2}}\right)$.

The equilibrium E^* is said to be locally asymptotically stable if all eigenvalues of (30) satisfy $| Arg(\xi_i) |> \frac{m\pi}{2}, \forall m \in (0, 1], i = 1, 2, 3$. One can then determine the stability of E^* by noting the signs of the coefficients A_i and discriminant $D(F)$ of the cubic polynomial $F(\xi)$ [13, 44]. The discriminant $D(F)$ of the cubic polynomial $F(\xi)$ is

$$\mathbf{D(F)} = -\begin{vmatrix} 1 & A_1 & A_2 & A_3 & 0 \\ 0 & 1 & A_1 & A_2 & A_3 \\ 3 & 2A_1 & A_2 & 0 & 0 \\ 0 & 3 & 2A_1 & A_2 & 0 \\ 0 & 0 & 3 & 2A_1 & A_2 \end{vmatrix}$$
$$= 18A_1A_2A_3 + (A_1A_2)^2 - 4A_3A_1^3 - 4A_2^3 - 27A_3^2$$

. Then the following theorem regarding local asymptotic stability of E^* of the system (10) is true [13, 20, 44].

Theorem 4 *(i) If $D(F) > 0$, $A_1 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$ then the interior equilibrium E^* is locally asymptotically stable for all $m \in (0, 1]$.*

(ii) If $D(F) < 0$, $A_1 \ge 0$, $A_2 \ge 0$, $A_3 > 0$ and $0 < m < \frac{2}{3}$ then the interior equilibrium E^* is locally asymptotically stable.

(iii) If $D(F) < 0$, $A_1 < 0$, $A_2 < 0$ and $m > \frac{2}{3}$ then the interior equilibrium E^* is unstable.

(iv) If $D(F) < 0$, $A_1 > 0$, $A_2 > 0$, $A_1A_2 = A_3$ and $0 < m < 1$ then the interior equilibrium E^* is locally asymptotically stable.

Then, I proceed to prove the global stability results of the interior equilibrium point $E^* = (x^*, y^*, z^*)$. The following lemma will be used in proving it.

Lemma 6 *[18] Let $x(t) \in \mathbb{R}_+$ be a continuous and derivable function. Then for any time instant $t > t_0$*

$${}^{c}_{t_0}D^m_t\left[x(t) - x^* - x^* ln\frac{x(t)}{x^*}\right] \le \left(1 - \frac{x^*}{x(t)}\right) {}^{c}_{t_0}D^m_t x(t), x^* \in \mathbb{R}_+, \forall m \in (0, 1].$$

Theorem 5 *The interior equilibrium $E^* = (x^*, y^*, z^*)$ of system (10) is globally asymptotically stable for any $m \in (0, 1]$ if*

$$(i) \quad \frac{2sy^*}{a(x^{*2}+a)} + \frac{s}{2a^2} - 1 < 0,$$

$$(ii) \quad \frac{cx^* - b(x^{*2}+a)}{a\beta d} + \frac{s}{2a^2} + \frac{1}{2}\Big(\frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)}\Big) < 0,$$

$$(iii) \quad \frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)} < 0,$$

where $\alpha = \frac{1}{b^2(\beta+\frac{\beta}{4b}+r)} > 0.$

Proof: Let us consider the Lyapunov function

$$V(x,y,z) = \Big(x - x^* - x^* ln\frac{x}{x^*}\Big) + \frac{(x^{*2}+a)}{a\beta}\Big(y - y^* - y^* ln\frac{y}{y^*}\Big) + (y^*+r)\Big(z - z^* - z^* ln\frac{z}{z^*}\Big).$$

It is easy to see that $V = 0$ only at $(x,y,z) = (x^*,y^*,z^*)$ and $V > 0$ whenever $(x,y,z) \neq (x^*,y^*,z^*)$. Considering the $m-th$ order fractional derivative of $V(x,y,z)$ along the solutions of (10), I have

$$ {}_0^cD_t^m V(x,y,z) = {}_0^cD_t^m\Big(x - x^* - x^* ln\frac{x}{x^*}\Big) + \frac{(x^{*2}+a)}{a\beta}{}_0^cD_t^m\Big(y - y^* - y^* ln\frac{y}{y^*}\Big) + (y^*+r){}_0^cD_t^m\Big(z - z^* - z^* ln\frac{z}{z^*}\Big).$$

Using Lemma (6) and making some algebraic manipulations, I have

$$\begin{aligned}
&{}_0^cD_t^m V(x,y,z) \\
&\leq \frac{(x-x^*)}{x}{}_0^cD_t^m x(t) + \frac{(x^{*2}+a)}{a\beta}\frac{(y-y^*)}{y}{}_0^cD_t^m y(t) + (y^*+r)\frac{(z-z^*)}{z}{}_0^cD_t^m z(t) \\
&= (x-x^*)\Big[(1-x) - \frac{sy}{x^2+a}\Big] + \frac{(x^{*2}+a)}{a}(y-y^*)\Big[\frac{sx}{x^2+a} - \frac{b}{\beta} - \frac{z}{\beta(d+y)}\Big] + (y^*+r)(z-z^*)\Big[pz - \frac{qz}{y+r}\Big] \\
&= (x-x^*)\Big[(x^*-x) + \frac{sy^*}{x^{*2}+a} - \frac{sy}{x^2+a}\Big] \\
&\quad + \frac{(x^{*2}+a)}{a}(y-y^*)\Big[\frac{sx}{x^2+a} - \frac{sx^*}{x^{*2}+a} + \frac{z^*}{\beta(d+y^*)} - \frac{z}{\beta(d+y)}\Big] \\
&\quad + z(y^*+r)(z-z^*)\Big[\frac{q}{y^*+r} - \frac{q}{y+r}\Big] \\
&= -(x-x^*)^2 + s(x-x^*)\Big[\frac{y^*(x^2+a) - y(x^{*2}+a)}{(x^2+a)(x^{*2}+a)}\Big] \\
&\quad + \frac{(x^{*2}+a)}{a}s(y-y^*)\Big[\frac{x(x^{*2}+a) - x^*(x^2+a)}{(x^2+a)(x^{*2}+a)}\Big] \\
&\quad + \frac{(x^{*2}+a)}{a\beta}(y-y^*)\Big[\frac{d(z^*-z) + \{y^*(z^*-z) + z^*(y-y^*)\}}{(y+d)(y^*+d)}\Big] \\
&\quad + \frac{qz}{y+r}(y-y^*)(z-z^*) \\
&\leq \Big[\frac{sy^*(x+x^*)}{(x^2+a)(x^{*2}+a)} - 1\Big](x-x^*)^2 - \frac{sxx^*}{a(x^{*2}+a)}(x-x^*)(y-y^*) \\
&\quad + \frac{(x^{*2}+a)}{a\beta}\frac{z^*(y-y^*)^2}{(y+d)(y^*+d)} \\
&\quad + \Big[\frac{qz}{y+r} - \frac{x^{*2}+a}{a\beta(y+d)}\Big](y-y^*)(z-z^*) \\
&\leq \Big[\frac{2sy^*}{a(x^{*2}+a)} - 1\Big](x-x^*)^2 + \frac{z^*(x^{*2}+a)(y-y^*)^2}{a\beta d(y^*+d)} \\
&\quad + \frac{sxx^*}{a(x^{*2}+a)}\Big[\frac{(x-x^*)^2 + (y-y^*)^2}{2}\Big] \\
&\quad + \Big[\frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)}\Big]\Big[\frac{(y-y^*)^2 + (z-z^*)^2}{2}\Big] \\
&\leq \Big[\frac{2sy^*}{a(x^{*2}+a)} - 1\Big](x-x^*)^2 + \frac{(cx^* - b(x^{*2}+a))}{a\beta d}(y-y^*)^2 \\
&\quad + \frac{s}{a^2}\Big[\frac{(x-x^*)^2 + (y-y^*)^2}{2}\Big] \\
&\quad + \Big[\frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)}\Big]\Big[\frac{(y-y^*)^2 + (z-z^*)^2}{2}\Big] \\
&= \Big[\frac{2sy^*}{a(x^{*2}+a)} + \frac{s}{2a^2} - 1\Big](x-x^*)^2 + \Big[\frac{cx^* - b(x^{*2}+a)}{a\beta d} + \frac{s}{2a^2} + \\
&\quad \frac{1}{2}\Big(\frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)}\Big)\Big](y-y^*)^2 \\
&\quad + \frac{1}{2}\Big[\frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)}\Big](z-z^*)^2.
\end{aligned}$$

One can note that ${}_0^cD_t^m V(x,y,z) \leq 0, \forall (x,y,z) \in \mathbb{R}_+^3$ if each coefficient of $(x-x^*)^2$, $(y-y^*)^2$ and $(z-z^*)^2$ are negative, giving the conditions

$$(i) \quad \frac{2sy^*}{a(x^{*2}+a)} + \frac{s}{2a^2} - 1 < 0,$$

$$(ii) \quad \frac{cx^* - b(x^{*2}+a)}{a\beta d} + \frac{s}{2a^2} + \frac{1}{2}\Big(\frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)}\Big) < 0,$$

$$(iii) \quad \frac{q}{4br\alpha(q-p(\beta+\frac{\beta}{4b}+r))} - \frac{x^{*2}+a}{a\beta(\beta+\frac{\beta}{4b}+d)} < 0.$$

Here ${}_0^cD_t^m V(x,y,z) = 0$ implies that $(x,y,z) = (x^*,y^*,z^*)$. Therefore, the only invariant set on which ${}_0^cD_t^m V(x,y,z) = 0$ is the singleton set $\{E^*\}$. Then, using Lemma (4.6) in [19], it follows that the interior equilibrium E^* is global asymptotically stable for any $m \in (0, 1]$. Hence the theorem is proven.

This global stability result is independent of fractional order m and it is also true for integer order ($m = 1$).

5. Numerical Simulations

In this section, I perform extensive numerical computations of the fractional order system (9) for different fractional values of

m $(0 < m < 1)$ and also for $m = 1$. I use Adams-type predictor corrector method (PECE) for the numerical solution of system (9). It is an effective method to give numerical solutions of both linear and nonlinear FODE [45, 46]. I first replace our system (9) by the following equivalent fractional integral equations:

$$\begin{aligned} X(T) &= X(0) + D_T^{-m}[a_0 X - b_0 X^2 - \frac{v_0 XY}{d_0 + X^2}], \\ Y(T) &= Y(0) + D_T^{-m}[-a_1 Y + \frac{v_1 XY}{d_1 + X^2} - \frac{v_2 YZ}{d_2 + Y}], \quad (31) \\ Z(T) &= Z(0) + D_T^{-m}[c_3 Z^2 - \frac{v_3 Z^2}{d_3 + Y}]. \end{aligned}$$

and then apply the PECE (Predict, Evaluate, Correct, Evaluate) method.

Several examples are presented to illustrate the analytical results obtained in the previous section. Specially, our main objective is to explore the possibility of dynamical behavior of the fractional order system (9) by depending on the sensitive parameter and as well as the fractional order by keeping others parameters unchanged. To understand the effect of fractional order on the system dynamics, I varied m in its range $0 < m < 1$. I also plotted the solutions for $m = 1$, whenever necessary, to compare the solution of fractional order system with that of integer order. In numerical simulations, Initial values are indicated with stars and equilibrium points are denoted by red circles.

Example 1. In this example, here the parameter values are chosen as $b_0 = 0.075$, $a_1 = 0.105$, $d_1 = d_2 = 10.0$, $d_3 = 20.0$, $v_0 = 1.0$, $v_1 = 2.0$, $v_2 = 0.405$, $v_3 = 1.0$ and the initial condition $(1.2, 1.2, 1.2)$. All the parameters are taken from [34]. The bifurcation diagram with respect to sensitive parameters a_0 and c_3 is shown in Fig. 1 for different fractional order $m = 0.95, 0.75$ and the standard order $m = 1$. For the standard order $m = 1$, it is observed that the system (9) approaches to chaos via period doubling bifurcation for $a_0 \in (0.25, 0.5)$ and $c_3 = 0.047$ (see Fig. 2(a)). It is interesting to note that the bifurcation disappears slowly with the decreasing of fractional order m (see Figs. 2(b) and 2(c)). One can note that for lower memory (i.e. for higher value of fractional order derivative m), system shows complex dynamics whereas for higher memory (i.e. for lower value of fractional order derivative m), system shows more simpler dynamics than the previous one.

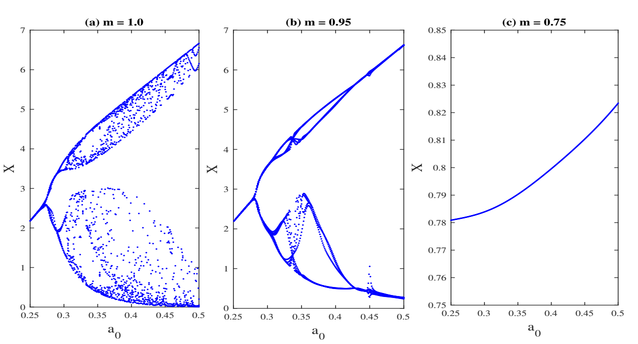

Figure 2. Bifurcation diagram of system (9) for the X population with respect to a_0 in $(0.25, 0.5)$ with different fractional orders $m = 0.95,\ 0.75$ (Fig. 2(b) and 2(c)) and integer order $m = 1$ (Fig. 2(a)). Here $b_0 = 0.075$, $a_1 = 0.105$, $d_1 = d_2 = 10.0$, $d_3 = 20.0$, $v_0 = 1.0$, $v_1 = 2.0$, $v_2 = 0.405$, $v_3 = 1.0$ with $c_3 = 0.047$.

Example 2. Here I fixed $a_0 = 0.47$ (say) and varying $c_3 \in$

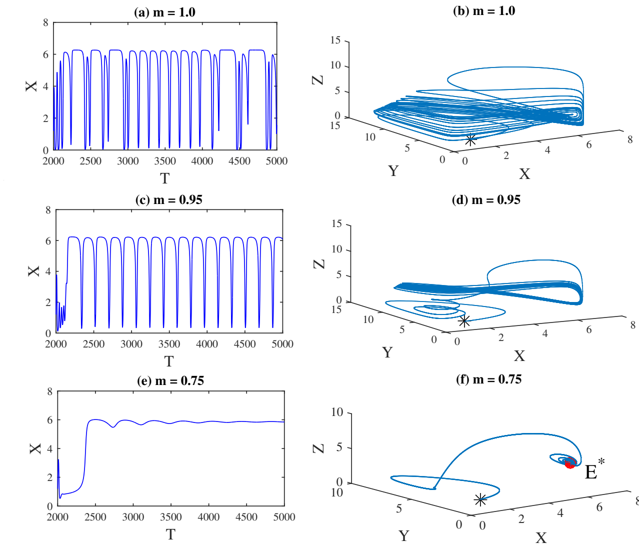

Figure 3. The trajectory and phase portrait of system (9) with different fractional orders $m = 0.95,\ 0.75$ (Fig. 3(c) - 3(f)) and integer order $m = 1$ (Fig. 3(a) - 3(b)). I observe that unstable behavior of our system changes to stability with decreasing of fractional order m. All the parameters are same as in example 1 with $a_0 = 0.47$ and $c_3 = 0.047$.

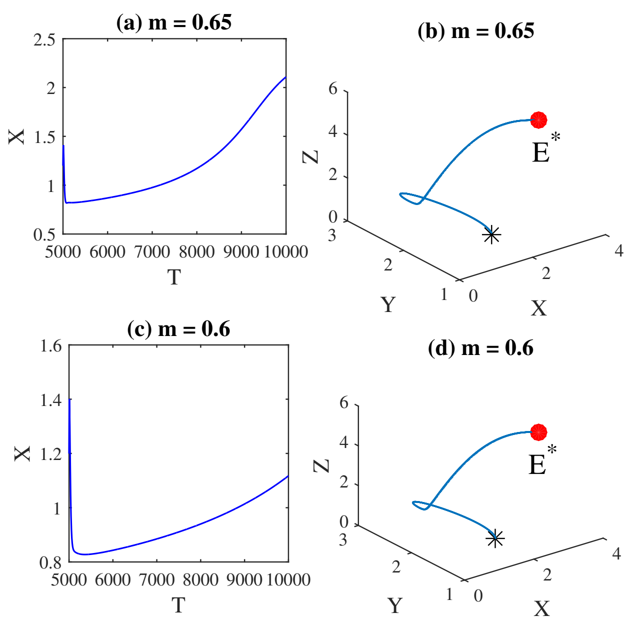

Figure 4. The trajectory and phase portrait of system (9) with different fractional orders $m = 0.65,\ 0.60 < \frac{2}{3}$ (Fig. 4(a) - 4(d)). I observe that the solution converges to interior equilibrium point for any values of $m < \frac{2}{3}$. It reaches to equilibrium value more slowly as the value of m becomes smaller. All the parameters are same as in example 1 with $a_0 = 0.27$ and $c_3 = 0.047$.

Example 3. Keeping c_3 unaltered, here I choose a smaller value of $a_0 = 0.27$ (say) and remaining all parameters are taken from example 1. Initial values are indicated with stars and equilibrium values are denoted by red circles in the figure. Step size for all simulations is considered as 0.05. Using the above parameter set, I first verify the existence criteria of E^*. Here I observe $y^* - \frac{a}{s} = -1.4644 < 0$, $b - \frac{cx^*}{a + x^{*2}} = -0.7582 < 0$ and $v_3 - c_3 d_3 = 0.06 > 0$. Hence $E^* = (2.5772, 1.2766, 5.7002)$ exists in $\mathbb{R}^3_+$. Then compute $D(F) = -0.0084 < 0$, $A_1 = 0.4033 > 0$, $A_2 = 0.0689 > 0$, $A_3 = 0.0221 > 0$. Thus, following Theorem (4)

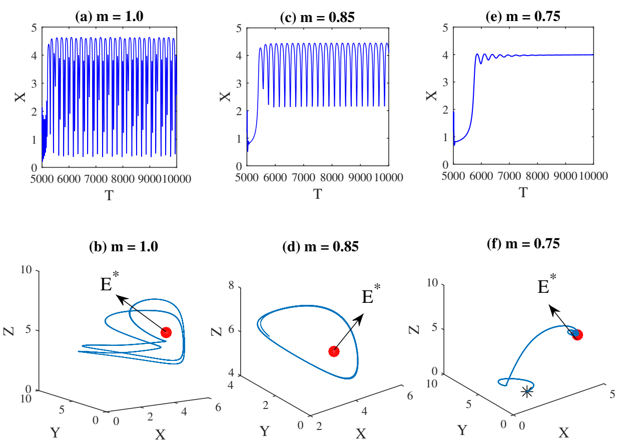

Figure 5. The trajectory and phase portrait of system (9) with different fractional orders $m = 0.85,\ 0.75$ (Figs. 5(c) - 5(f)) and for integer order $m = 1$ (Figs. 5(a) - 5(b)). I observe that the solution converges to interior equilibrium point for any values of $m < \frac{2}{3}$. It reaches to equilibrium value more slowly as the value of m becomes smaller. All the parameters are same as in example 1 with $a_0 = 0.35$ and $c_3 = 0.047$.

(ii), the interior equilibrium E^* should stable for $0 < m < \frac{2}{3}$. In Fig. 3, I plot the time series solutions and phase portrait of FDE system (9) with different values of $m = 0.65,\ 0.60 < \frac{2}{3}$. It shows that all populations remain stable for all values of $m < \frac{2}{3}$, though solutions reach to equilibrium value more slowly as the value of m becomes smaller (see Figs. 3(a) - 3(d)).

Again if I increase the value of $a_0 = 0.35$ and keeping all parameters unaltered as in example 1, I see that our system (9) exhibits 2-periodic limit cycle, 1-periodic limit cycle for higher values of fractional order $m = 0.85$ as well as for integer order $m = 1$ (see Figs. 5(a) - 5(d)). If I decrease the value of m, then limit cycle disappears and system becomes stable. Here I choose $m = 0.75$ and observe that solution converges to interior equilibrium point $E^* = (4.0150, 1.2766, 0.7816, 5.6362)$ (see Figs. 5(e) - 5(f)).

Example 4: To demonstrate the global stability of the interior equilibrium point E^*, I consider the parameter values $a_0 = 0.47$, $b_0 = 0.25$, $\upsilon_0 = 1.0$, $d_0 = d_1 = d_2 = 10.0$, $a_1 = 0.105$, $\upsilon_1 = 2.0$, $\upsilon_2 = 0.405$, $\upsilon_3 = 1.0$, $c_3 = 0.047$, $d_3 = 20.0$ and different initial points $(1.2, 1.2, 1.2)$, $(5.1, 2.1, 3)$, $(3, 1, 5)$, $(2, 5, 3.5)$, $(3, 1, 2)$, $(2.5, 5, 4)$, $(1.5, 5.5, 2)$, $(4.5, 5.5, 5)$. Initial values are indicated with stars and equilibrium values are denoted by red circles in the figure. Step size for all simulations is considered as 0.05. Using the above parameter set, I first verify the existence criteria of E^*. Here I observe $y^* - \frac{a}{s} = -3.8744 < 0$, $b - \frac{cx^*}{a+x^{*2}} = -0.2884 < 0$ and $\upsilon_3 - c_3 d_3 = 0.06 > 0$. Hence $E^* = (1.4589, 1.2766, 3.7751)$ exists in $\mathbb{R}^3_+$. With these parameter values, I verify that all conditions of Theorem (5) are satisfied as

$$(i)\quad \frac{2sy^*}{a(x^{*2}+a)} + \frac{s}{2a^2} - 1 = -0.8084 < 0,$$

$$(ii)\quad \frac{cx^* - b(x^{*2}+a)}{a\beta d} + \frac{s}{2a^2} + \frac{1}{2}\Big(\frac{q}{4br\alpha(q - p(\beta + \frac{\beta}{4b} + r))} - \frac{x^{*2}+a}{a\beta(\beta + \frac{\beta}{4b} + d)}\Big) = -0.0906 < 0,$$

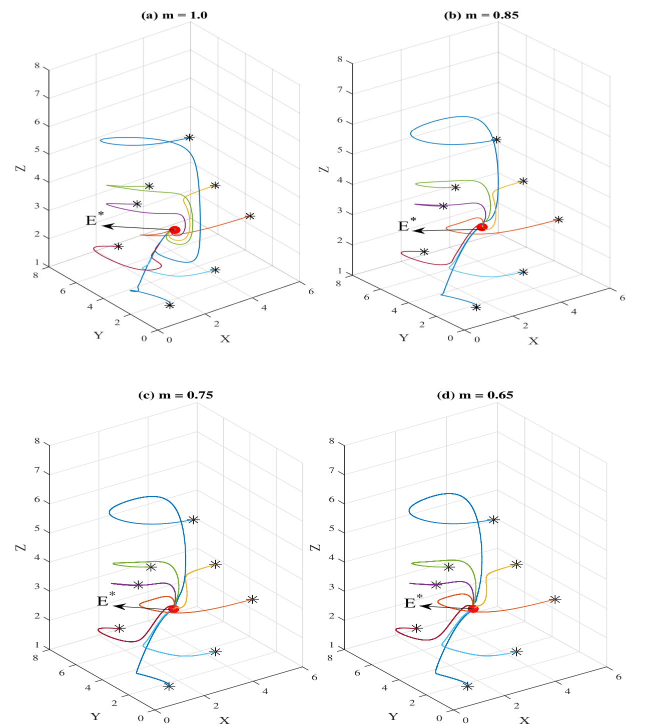

Figure 6. Trajectories with different initial values converge to the interior equilibrium point E^* for different values of m, indicating global stability of the equilibrium E^* when conditions of Theorem (5) are satisfied. All parameters are as in Fig. 1 except $b_0 = 0.25$.

$$(iii)\quad \frac{q}{4br\alpha(q - p(\beta + \frac{\beta}{4b} + r))} - \frac{x^{*2}+a}{a\beta(\beta + \frac{\beta}{4b} + d)} = -0.2623 < 0.$$

where $\alpha = \frac{1}{b^2(c + \frac{c}{4b} + r)} = 0.6330 > 0$. Fig. 5 demonstrates that solutions starting from different initial values converge to the equilibrium point $E^* = (1.4589, 1.2766, 3.7751)$ for different fractional orders, $m = 0.65, 0.75, 0.85$, and also for the integer order, $m = 1$, depicting the global stability of the interior equilibrium point for fractional order as well as integer order (see Figs. 6(a) - 6(d)).

6. Conclusions

In this paper, I generalize the study of integer order three species food chain model [34] with simplified Holling type IV functional response by using the memory effect related to fractional order derivative. Following fractional order Caputo derivative approach, here I first convert the integer order differential equations of the three species predator-prey model (4) to the fractional order differential equations and modeled the system (9), which allow us to consider memory effects. I investigated some qualitative behaviours of the system (9) like existence and uniqueness, non-negativity and boundedness which are systematically discussed in $\mathbb{R}^3_+$. Local stability criteria of the different equilibrium points have been discussed for fractional order system. Global stability of the interior equilibrium point have been only discussed. I defined suitable Lyapunov function to prove that the interior equilibrium is globally asymptotically stable if the system parameters satisfy some conditions. In such a case, the system does not show any complicated dynamics like chaos, indicating its global stability for any fractional order $0 < m < 1$. This is more reinforced by the fact that solutions initiating from biologically feasible arbitrary initial points converge to the interior equilibrium point. To confirm the analytical results of our system, numerical simulation is performed for different sets of biologically feasible parameter values. Simulation results also agree perfectly with the analytical results. Numerically it has been observed that the fractional order system (9) shows more complex dynamics, like chaos, bifurcation for higher memory as the fractional order becomes larger and shows more simpler dynamics for lower memory as the order m decreases. Specially, due to memory effect, it becomes stable for lower value of m. Moreover, dynamics of the fractional-order system not only depends on system parameters but also depends on fractional order m. Reader can note that for lower memory or for memory less system (i.e. for higher value of fractional order derivative m), system shows complex dynamics like chaos, bifurcation etc. whereas for higher memory or for memory system (i.e. for lower value of fractional order derivative m), system shows more simpler dynamics which actually shows the effect of memory for fractional order dynamical systems.

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