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The Structure of Center of Semiring

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Abstract

The center plays a fundamental role in group theory, algebraic geometry, ring theory, and semiring theory. Considerable research has been devoted to the study of centers in semiring, yielding important insights into the development of semiring theory. This article introduces the notion of the center of a semiring and aims to investigate its structural properties within the framework of semiring theory. Additionally, several algebraic characterizations of center semiring are examined.

Keywords: *Semiring; Center of semiring; Matrix semiring.*

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1. INTRODUCTION

In recent years, there has been a substantial increase in interest in the study of partially ordered and totally ordered algebraic structures such as semigroups, groups, semirings, semimodules, rings, and fields. A significant portion of ring theory has been extended to general semirings. Some mathematicians even argue that semirings are the more fundamental algebraic structures, and that restricting attention to rings is analogous to specializing to algebra over the complex numbers.

The concept of a semiring was introduced by Vandiver in 1934. However, semirings had already established their significance in mathematics well before this period. The development of semiring theory has been progressing steadily since the 1950s. The theories of rings and semigroups have a substantial influence on the advancement of semirings and ordered semirings

In the literature, a semiring is a non-empty set S together with two binary operations "+" and "." (usually denoted by juxtaposition) such that $(S, +)$ is a commutative semigroup and $(S, \cdot)$ is a semigroup, which are connected by ring-like distributivity. So semirings which provide a common generalization of rings and distributive lattices appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics. In the study of semirings and their representations, researchers employ methods and techniques from ring theory and lattice theory, along with a wide range of tools from categorical and universal algebra.

A semiring S is called a semiring with zero element '0' if $a+0 = 0+a = a$ and $0.a = a.0 = 0$ for all $a \in S$. A semiring S is called a semiring with identity 1 if $1 \cdot a = a \cdot 1 = a$ for all $a \in S$.

A semiring may or may not have a zero and an identity element.

Throughout this paper a semiring $(S, +, .)$ with zero element '0' and identity element '1' is considered.

Let $(S, +, .)$ and $(T, +, .)$ be two semirings. Then a mapping $f : S \longrightarrow T$ is said to be a semiring homomorphism [1] of S into T if $f(x + y) = f(x) + f(y)$, and $f(xy) = f(x)f(y)$, for all $x, y \in S$. An injective homomorphism is called a monomorphism, a surjective homomorphism is called an epimorphism and a bijective homomorphism is called an isomorphism.

The notion of center $Z(R)$ of a ring R is a substructure consisting of the elements x such that $xy = yx$ for all y in R. Moreover $Z(R)$ is a subring of the ring R but not necessarily an ideal of R. Several authors studied various center-like subsets for rings and under certain natural conditions. However, the study of centers within the framework of semirings has received relatively little attention so far.

Some recent works on center-like subsets of rings can be found in [2], [3], [4] . In 2006, M.K. Sen et al. defined [5] the term "Center of Semiring" to characterize a unique class of elements in a semiring. In [6], the notion of the Birkhoff centre of a c-semiring was introduced, and in [7], its structure was studied. The purpose of this paper is to investigate many results on center of a semiring which are analogous to the same direction in ring theory.

2. CENTER OF A SEMIRING S

In this section, some examples and basic results are provided, as they are useful for subsequent results in the next sections.

Definition 2.1 *Let S be a semiring. A subset $Z(S)$ of a semiring S is called a center of S which is defined by $Z(S) = \{a \in S : ab = ba$ for all $b \in S\}$.*

Example 2.2 *Consider $(\mathbb{N}, \oplus, \odot)$ is a semiring , where $a \oplus b = max\{a, b\}$ and $a \odot b = min\{a, b\}$. Then $Z(\mathbb{N}) = \mathbb{N}$.*

Example 2.3 *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables:*

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S, +, \cdot)$ is a semiring and $Z(S) = \{0, x, 1\} = S$.

Example 2.4 *Consider $S = \{0, x, y, 1\}$. Define the operations "+" and "." on S by the following tables :*

+	0	x	y	1
0	0	x	y	1
x	x	x	y	1
y	y	y	y	1
1	1	1	1	1

.	0	x	y	1
0	0	0	0	0
x	0	x	x	x
y	0	x	y	y
1	0	x	y	1

Then $(S, +, \cdot)$ is a semiring and $Z(S) = \{0, x, y, 1\} = S$.

Theorem 2.5 *The center of a semiring S is a subsemiring of S.*

Proof: Since S is a semiring with zero element 0, it follows that $0 \in Z(S)$. Thus, the center of a semiring S is non-empty. Let $x, y \in Z(S)$. Then it shows that

$$x + y \in Z(S) \text{ and } xy \in Z(S).$$

For any $k \in S$,

$$(x + y)k = xk + yk = kx + ky$$

since $x \in Z(S)$ and $y \in Z(S)$

$$= k(x + y).$$

Therefore, $x + y \in Z(S)$.

Similarly, for any $k \in S$,

$$xyk = xky \text{ since } y \in Z(S)$$
$$= kxy \text{ since } x \in Z(S).$$

Therefore, $xy \in Z(S)$. Consequently, $Z(S)$ is closed under addition and multiplication and hence forms a subsemiring of S.

3. SOME PROPERTIES OF $Z(S)$ OF SEMIRING S

In this section, some elementary properties of $Z(S)$ for the semiring S are discussed. The discussion begins by exploring the isomorphism property of $Z(S)$ of the semiring S.

Theorem 3.1 *If two semirings S_1 and S_2 are isomorphic, then their centers $Z(S_1)$ and $Z(S_2)$ are isomorphic.*

Proof: Consider two semirings S_1 and S_2 which are isomorphic. Then there is an isomorphism

$$f : S_1 \longrightarrow S_2.$$

Let $x \in Z(S_1)$. Then for any $s_1 \in S_1$,

$$xs_1 = s_1x.$$

Let $f(x) = y$, where $y \in S_2$.

Since f is an isomorphism, for any $s_2 \in S_2$, there exists $s_1 \in S_1$ such that $f(s_1) = s_2$. Thus

$$ys_2 = f(x)f(s_1) = f(xs_1) = f(s_1x) = f(s_1)f(x) = s_2y,$$

since $x \in Z(S_1)$.

Therefore, $y \in Z(S_2)$, which shows that $f(Z(S_1)) \subseteq Z(S_2)$. Again, let $b \in Z(S_2)$. Then $b = f(a)$ where $a \in S_1$.

Since f is an isomorphism, for any $y \in S_2$, there exists $x \in S_1$ such that $y = f(x)$. Since $b \in Z(S_2)$, it follows that $by = yb$. Now

$$\begin{aligned} & by = yb \\ \Longrightarrow\ & f(a)f(x) = f(x)f(a) \\ \Longrightarrow\ & f(ax) = f(xa) \\ \Longrightarrow\ & ax = xa, \end{aligned}$$

since f is an isomorphism. This implies that $a \in Z(S_1)$.

Therefore, $b = f(a) \in f(Z(S_1))$. Thus $Z(S_2) \subseteq f(Z(S_1))$, and together with the earlier inclusion, it follows that $Z(S_2) = f(Z(S_1))$. So,

$$g = f_{|_{Z(S_1)}} : Z(S_1) \longrightarrow Z(S_2)$$

is well defined and it is an isomorphism from $Z(S_1)$ onto $Z(S_2)$.

The following example shows that the converse of the above theorem 3.1 is not true i.e. if $Z(S_1)$ and $Z(S_2)$ are isomorphic then S_1 and S_2 may not be isomorphic, in general.

Example 3.2 *$Z(\mathbb{Z}_0^+) = \{0\}$ and $Z(\mathbb{R}_0^+) = \{0\}$. But $\mathbb{Z}_0^+$ and $\mathbb{R}_0^+$ are not isomorphic.*

Theorem 3.3 *Let S and S' be two semiring. If $f : S \to S'$ is a monomorphism, then $f(Z(S)) = Z(f(S))$.*

Proof: Suppose $x \in f(Z(S))$. Then $x = f(y)$ for some $y \in Z(S)$. Then it is necessary to show that

$$f(y)s = sf(y)$$

for all $s \in f(S)$. Now for any $s \in f(S)$, it follows that

$$f(y)s = f(y)f(r) = f(yr) = f(ry) = f(r)f(y) = sf(y).$$

Therefore, $x = f(y) \in f(Z(S))$. Hence $f(Z(S)) \subseteq Z(f(S))$.

Again, let $x' \in Z(f(S))$. Then $x' = f(r')$ for some $r' \in S$. The next step is to show that $r' \in Z(S)$. Since $x' \in Z(f(S))$, for any $f(s) \in f(S)$, it follows that

$$\begin{aligned} & x'f(s) = f(s)x' \\ \Longrightarrow\ & f(r')f(s) = f(s)f(r') \\ \Longrightarrow\ & f(r's) = f(sr') \\ \Longrightarrow\ & r's = sr', \end{aligned}$$

since f is a monomorphism . Therefore, $r' \in Z(S)$. Thus $Z(f(S)) \subseteq f(Z(S))$. These two inclusions conclude that $Z(f(S)) = f(Z(S))$.

Theorem 3.4 *Let S be a commutative semiring with identity element 1. Then $a \in Z(S)$ if and only if $aI_n \in Z(M_n(S))$.*

Proof: Let $a \in Z(S)$. Then $ab = ba$ for all $b \in S$. To show that $aI_n \in Z(M_n(S))$.
Now

$$
\begin{aligned}
(aI_n)B &= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \\
&= \begin{pmatrix} ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & ab_{22} & \cdots & ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & ab_{nn} \end{pmatrix} \\
&= \begin{pmatrix} b_{11}a & b_{12}a & \cdots & b_{1n}a \\ b_{21}a & b_{22}a & \cdots & b_{2n}a \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}a & b_{n2}a & \cdots & b_{nn}a \end{pmatrix} (\text{since } a \in Z(S)) \\
&= \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \\
&= B(aI_n).
\end{aligned}
$$

This implies that $aI_n \in Z(M_n(S))$.
Conversely, suppose that $aI_n \in Z(M_n(S))$. It remains to show that $a \in Z(S)$. For any $x \in S$, let $B = \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}$. Since $aI_n \in Z(M_n(S))$, it follows that $(aI_n)B = B(aI_n)$. Comparing both sides, yields $ax = xa$. Since x is arbitrary, it follows that $a \in Z(S)$.

Theorem 3.5 *If S_1 and S_2 are two semirings, then $Z(S_1 \times S_2) = Z(S_1) \times Z(S_2)$.*

Proof: Let S_1 and S_2 be two semirings with zero elements 0_{S_1} and 0_{S_2} respectively. Suppose $z \in Z(S_1 \times S_2)$. Then $z = (x, y) \in S_1 \times S_2$ and for any $(a, b) \in S_1 \times S_2$,

$$
\begin{aligned}
& (x, y)(a, b) = (a, b)(x, y) \\
\Longrightarrow \quad & (xa, yb) = (ax, by).
\end{aligned}
$$

Comparing both sides, yields

$$xa = ax, \text{ and } yb = by.$$

This implies that $xa = ax$ for all $a \in S_1$ and $yb = by$ for all $b \in S_2$. Thus, it follows that $x \in Z(S_1)$ and $y \in Z(S_2)$. Therefore,

$$z = (x, y) \in Z(S_1) \times Z(S_2)$$

and hence

$$Z(S_1 \times S_2) \subseteq (S_1) \times Z(S_2). \tag{1}$$

For reverse part, let $(a, b) \in Z(S_1) \times Z(S_2)$. This implies that $a \in Z(S_1)$ and $b \in Z(S_2)$.

Consequently, every $x \in S_1$ satisfies $ax = xa$, while each $y \in S_2$ satisfies $by = yb$. Now,

$$(a, b)(x, y) = (ax, by) = (xa, yb) = (x, y)(a, b),$$

since $a \in Z(S_1)$ and $b \in Z(S_2)$. This implies that

$$(a, b) \in Z(S_1 \times S_2)$$

and hence

$$Z(S_1) \times Z(S_2) \subseteq Z(S_1 \times S_2) \tag{2}$$

From (1) and (2), it follows that $Z(S_1 \times S_2) = Z(S_1) \times Z(S_2)$.

4. APPLICATION OF CENTER SEMIRING

The notion of the center in semiring theory developed as a generalization of the corresponding concept in ring theory. The center plays a vital role in the study of semirings and other algebraic structures by helping to analyze their fundamental properties. It has numerous important applications in algebra and related fields. A semiring is commutative precisely when it is equal to its center, making the center an effective tool for identifying and understanding non-commutativity. Moreover, this concept finds significant applications in practical areas such as cryptography, coding theory, and communication.

5. CONCLUSION

In this paper, it has been shown that the center of a semiring is itself a subsemiring. The study of the algebraic structures associated with centers of semiring is therefore of considerable interest. This work also encourages further investigation into various types of centers in semirings. Since semirings admit many special kinds of centers, the author intends to explore several of them in future research. Moreover, there are numerous open problems related to these special centers and it is hoped that solutions to some of these problems will be obtained. It is also anticipated that future research will extend to other algebraic structures with significant practical applications, such as cryptography, coding theory, and communication systems.

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